

# THE FUNDAMENTAL GROUP OF THE COMPLEMENT OF THE BRANCH CURVE OF THE HIRZEBRUCH SURFACE $F_1$

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**ABSTRACT.** Given a projective surface and a generic projection to the plane, the braid monodromy factorization (and thus, the braid monodromy type) of the complement of its branch curve is one of the most important topological invariants ([10]), stable on deformations. From this factorization, one can compute the fundamental group of the complement of the branch curve, either in  $\mathbb{C}^2$  or in  $\mathbb{CP}^2$ . In this article, we show that these groups, for the Hirzebruch surface  $F_{1,(a,b)}$ , are almost-solvable. That is - they are an extension of a solvable group, which strengthen the conjecture on degeneratable surfaces (see [13]).

**keywords:** Hirzebruch surfaces, degeneration, generic projection, branch curve, braid monodromy, fundamental group, classification of surfaces.

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## 1. INTRODUCTION

In the study of smooth algebraic surfaces of degree  $n$ , which are embedded in  $\mathbb{CP}^N$ , one can consider the surface  $X$  as a branched cover of  $\mathbb{CP}^2$ . In this case the branch locus,  $S_X$  in  $\mathbb{CP}^2$ , plays a crucial role. It is, in general, singular and, if the projection  $X \rightarrow \mathbb{CP}^2$  is generic, the singularities are nodes and cusps. The significance of  $S_X$  (or of  $S \subset \mathbb{C}^2 \subset \mathbb{CP}^2$ , a generic affine portion of  $S_X$ ) arises when studying equivalence class of the braid monodromy factorization of the branch curve  $S_X$  (which is known to be the BMT invariant of the surface  $X$ ; see [13]). From this factorization one can induce the fundamental groups  $\overline{G} = \pi_1(\mathbb{CP}^2 - S_X)$  or  $G = \pi_1(\mathbb{C}^2 - S)$ , which are stable on deformations. That is, if two surfaces have different fundamental groups, then they are not deformation equivalent. For surfaces  $X, Y$  denote  $X \stackrel{G}{\simeq} Y \Leftrightarrow G_X = G_Y$  and  $\overline{G}_X = \overline{G}_Y$ ;  $X \stackrel{Diff}{\simeq} Y \Leftrightarrow X$  is diffeomorphic to  $Y$ ;  $X \stackrel{Def}{\simeq} Y \Leftrightarrow X$  is deformation equivalent to  $Y$ ; and  $X \stackrel{BMT}{\simeq} Y \Leftrightarrow X$  and  $Y$  has the same BMT invariant.

It turns out that  $X \stackrel{Def}{\simeq} Y \Rightarrow X \stackrel{G}{\simeq} Y$  but the inverse direction is not correct (see [11]); and  $X \stackrel{Def}{\simeq} Y \Rightarrow X \stackrel{BMT}{\simeq} Y \Rightarrow X \stackrel{Diff}{\simeq} Y$  (and again - the inverse directions are not correct; see [5],[11]).

In this article, we take  $X$  to be the Hirzebruch surface  $F_1$ ; this surface is the projectivization of the line bundle  $\mathcal{O}_{\mathbb{CP}^1}(1) \oplus \mathcal{O}_{\mathbb{CP}^1}$ . We then embed it in  $\mathbb{CP}^n$  with respect to the linear system  $|aC + bE_0|$ , where  $C, E_0$  generate the Picard group of  $F_1$ ,  $b > 1, a \geq 1$ . We show that  $G$  and  $\overline{G}$  can be computed when  $X = F_{1,(a,b)}$ , which is the image of  $F_1$  after the embedding w.r.t. the above linear system.

It is conjectured ([13]) that  $G$  and  $\overline{G}$  are almost solvable in a large family of surfaces: that is, these groups are extensions of a solvable group by the symmetric group. So far, it was proven for  $V_p$  (the Veronese surface; [14]) and  $X_{p,q}$  (the double-double covering of  $\mathbb{CP}^1 \times \mathbb{CP}^1$ ; [2]).

Our main result proves that  $X = F_{1,(a,b)}$  ( $b > 1, a \geq 1$ ) satisfies the conjecture. In particular, there exists a series

$$1 \triangleleft A_1 \triangleleft A_2 \triangleleft A_3 \triangleleft G$$

s.t.

$$G/A_3 \simeq S_{2ab+b^2},$$

$$A_3/A_2 \simeq \mathbb{Z},$$

$$A_2/A_1 \simeq (\mathbb{Z}_{b-2a})^{2ab+b^2-1}$$

$$A_1 \simeq \begin{cases} \mathbb{Z}_2 & b \text{ even, } a \text{ odd} \\ 1 & \text{otherwise} \end{cases}$$

and a series

$$1 \triangleleft \overline{A}_1 \triangleleft \overline{A}_2 \triangleleft \overline{A}_3 \triangleleft \overline{G}$$

where

$$\overline{G}/\overline{A}_3 = G/A_3,$$

$$\overline{A}_3/\overline{A}_2 \simeq \mathbb{Z}_m, \quad m = 3ab - a - b + \frac{3b^2 - 3b}{2},$$

$$\overline{A}_2/\overline{A}_1 = A_2/A_1,$$

$$\overline{A}_1 = A_1.$$

As noted, the significance of this article lies in the fact that  $G$  and  $\overline{G}$  are determined by the deformation type, since they are stable under deformation of the surface. Thus, computing  $G$  and  $\overline{G}$  explicitly (and the series of groups derived from them) can help us distinguish between non-deformation equivalent Hirzebruch surfaces.

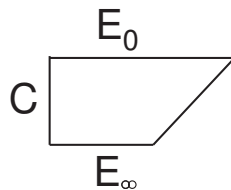
Another important aspect of this article is the fact that it gives a general approach and another example of how to compute and deal with the fundamental groups  $G$  and  $\overline{G}$ . So far, only a few examples of calculating these groups were presented (see [8], [15]), and most of the calculations dealt with the Galois cover of such a degeneratable surface; especially with finding the fundamental group of this Galois cover (see [9], [3]). Calculating  $G$  and  $\overline{G}$  is another step in understanding the whole structure of these groups with respect to surfaces which can be degenerated.

## 2. HIRZEBRUCH SURFACES AND THEIR DEGENERATIONS

The Hirzebruch surfaces  $F_k$  (for  $k \geq 0$ ) are given by the equation  $x_1 y_1^k = x_2 y_2^k$  in  $\mathbb{CP}^1 \times \mathbb{CP}^1$ . However, the construction these days is as follows: the  $k$ -th Hirzebruch surface is the projectivization of the vector bundle  $\mathcal{O}_{\mathbb{CP}^1}(k) \oplus \mathcal{O}_{\mathbb{CP}^1}$ .

Let  $\sigma$  be a holomorphic section of  $\mathcal{O}_{\mathbb{CP}^1}(k)$ , and let  $E_0 \subset F_k$  denote the image of the section  $(\sigma, 1)$  of  $\mathcal{O}_{\mathbb{CP}^1}(k) \oplus \mathcal{O}_{\mathbb{CP}^1}$ . The curve  $E_0$  is called a *zero section* of  $F_k$ . All zero sections are homologous and hence define a divisor class which is independent of choice of  $\sigma$ . Let  $C$  denote a fiber of  $F_k$ . The Picard group of  $F_k$  is generated by  $E_0$  and  $C$ . It is elementary that  $E_0^2 = k$ ,  $C^2 = 0$  and  $E_0 \cdot C = 1$ .

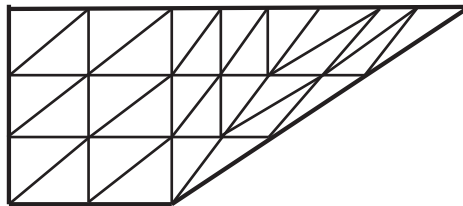
The surface  $F_0$  is the quadric  $\mathbb{CP}^1 \times \mathbb{CP}^1$ , and  $F_1$  is the blow-up of the plane  $\mathbb{CP}^2$ . For  $k > 0$ , the surface  $F_k$  contains a unique (irreducible) curve of negative self-intersection  $-k$ . This curve is a section of the bundle; it is denoted  $E_\infty$  and is called the *negative section* or the *section at infinity*. We mention that it can be contracted to an isolated normal singularity, the resulting normal surface being the cone over the rational normal curve of degree  $k$ . Zero sections are always disjoint to  $E_\infty$ . Schematically, we describe  $F_k$  as in Fig. 1.1.



(figure 1.1)

Let  $F_k$  be the  $k$ -th Hirzebruch surface. Let  $E_0$ ,  $E_\infty$ ,  $C$  be as in the Introduction. For  $a, b \geq 1$ , or for  $a = 0$  and  $k \geq 1$ , the divisor  $ac + bE_0$  on  $F_k$  is very ample and thus defines an embedding  $f_{|ac+bE_0|} : F_k \hookrightarrow \mathbb{CP}^N$ . Let  $F_{k(a,b)} = f_{|ac+bE_0|}(F_k) (\subseteq \mathbb{CP}^N)$ . For  $k > 0$ , the map  $f_{|0 \cdot C + bE_0|}$  collapses the section at infinity to a point, so  $F_{k(0,b)}$  is the image of the cone over the rational normal curve of degree  $k$  with respect to a suitable embedding.

In [9], a degeneration to a union of  $2ab + kb^2$  planes was constructed in the following configuration (in Fig. 1.2,  $k = 2$ ,  $a = 2$ ,  $b = 3$  was taken). Each triangle represents a plane and each inner edge represents an intersection line between planes.



(figure 1.2)

This degeneration is obtained using a technique developed by Moishezon-Robb-Teicher which they refer to as the D-construction. The D-construction is described (and prove to work) in [8]. Specific degeneration for the Hirzebruch surfaces using the D-construction is explained in [9, Section 2, Theorem 2.1.2]. The difference between the D-construction and other blow-up procedures for obtaining degenerations is that the D-construction can also be applied along a subvariety of codim 0 (see, for example, Step 2 below). The degeneration is obtained via the following steps.

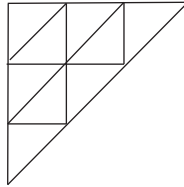
1. D-construction along  $C$  to get  $F_{0(a,b)} \cup F_{k(a-1,b)}$ .

2. D-construction along  $F_{0(1,b)}$  to get  $F_{0(1,b)} \cup F_{0(1,b)} \cup F_{k(a-2,b)}$ .
3. Induction on the second step to get  $\underbrace{F_{0(1,b)} \cup \cdots \cup F_{0(1,b)}}_{a \text{ times}} \cup F_{k(0,b)}$  (see [?]).
4. Degeneration of each  $F_{0(1,b)}$  to a union of  $2b$  planes in the following configuration (here  $b = 3$ ).



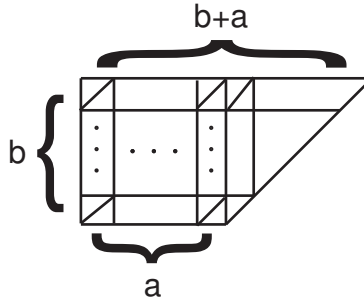
(figure 1.3)

5. D-construction on  $F_{k(0,b)}$  to get  $\underbrace{F_{1(0,b)} \cup \cdots \cup F_{0(1,b)} \cup F_{1(0,b)}}_{k \text{ times}}$   
( $F_{1(0,b)}$  is the Veronese surface  $V_b$ ).
6. Degeneration of each  $F_{1(0,b)}$  to a union of  $b^2$  planes in the following configuration (here  $b = 3$ ) :



(figure 1.4)

Note that in our case  $k = 1$ ; so we are looking at the surface  $F_{1(a,b)}$ . We now describe in greater detail the degenerated object and its branch curve using the degeneration described earlier.  $F_{1(a,b)}$  is degenerated to  $\tilde{F}_{1(a,b)}$  – a union of planes in the following configuration.



(figure 1.5)

Each triangle represents a plane; each inner edge represents an intersection line between planes. The number of the planes is  $2ab + b^2$ ; the number of intersection lines is  $3ab - a + \frac{3b}{2}(b-1)$ . We take a generic projection of  $\tilde{F}_{1(a,b)}$  onto  $\mathbb{CP}^2$  where each plane is projected onto  $\mathbb{CP}^2$ . The ramification curve of this projection is the union of lines. The singular points of the ramification curve are represented by the vertices. The branch curve of  $\tilde{F}_{1(a,b)} \rightarrow \mathbb{CP}^2$ , denoted by  $\tilde{S}_{(a,b)}$ , is the image of the union of lines and its singular points are the images of the vertices and the intersection points in  $\mathbb{CP}^2$  of the images of any two of the intersection lines. Special notations of the vertices and the edges of the complex in Fig. 1.5 (which represent  $\tilde{S}_{(a,b)}$ ) will be given in Section 4.

### 3. $B_n, \tilde{B}_n$ AND $\tilde{B}_n$ -GROUPS

The aim of this section is to introduce a few facts about  $B_n$  and a certain quotient of it, which will serve us in the next section.

**Definition 3.1.**  $B_n, S_n$ :

The braid group on  $n$  strings is

$$B_n = \left\{ x_1, \dots, x_{n-1} \mid \begin{array}{ll} [x_1, x_j] = 1 & |i - j| > 1 \\ \langle x_i, x_j \rangle = 1 & |i - j| = 1 \end{array} \right\}.$$

Recall that the permutation group is

$$S_n = \left\{ x_1, \dots, x_{n-1} \mid \begin{array}{ll} [x_1, x_j] = 1 & |i - j| > 1 \\ \langle x_i, x_j \rangle = 1 & |i - j| = 1 \end{array}, x_i^2 = 1 \right\}.$$

So,  $\exists$  homomorphism  $\varphi : B_n \rightarrow S_n$ . Denote by  $\delta$  the degree homomorphism  $\delta : B_n \rightarrow \mathbb{Z}$ ; denote  $P_n = \ker \varphi$ ,  $P_{n,0} = P_n \cap \ker \delta$ .

We now recall another definition of  $B_n$ .

Let  $D$  be a closed disk in  $\mathbb{R}^2$ ,  $K \subset \text{Int}(D)$ ,  $K$  finite,  $n = \#K$ . Recall that the braid group  $B_n[D, K]$  can be defined as the group of all equivalent diffeomorphisms  $\beta$  of  $D$  such that  $\beta(K) = K$ ,  $\beta|_{\partial D} = \text{Id}|_{\partial D}$ .

**Definition 3.2.**  $H(\sigma)$ , half-twist defined by  $\sigma$

Let  $a, b \in K$ , and let  $\sigma$  be a smooth simple path in  $\text{Int}(D)$  connecting  $a$  with  $b$  s.t.  $\sigma \cap K = \{a, b\}$ . Choose a small regular neighborhood  $U$  of  $\sigma$  contained in  $\text{Int}(D)$ , s.t.  $U \cap K = \{a, b\}$ . Denote by  $H(\sigma)$  the diffeomorphism of  $D$  which switches  $a$  and  $b$  by a counterclockwise 180 degree rotation and is the identity on  $D \setminus U$ . Thus it defines an element of  $B_n[D, K]$ , called *the half-twist defined by  $\sigma$* .

**Definition 3.3.**  $\tilde{B}_n$

Let  $\tilde{B}_n$  be the quotient of  $B_n$  by the following commutator,  $\tilde{B}_n = B_n / \langle [x_2, (x_2)_{x_1 x_3}] \rangle$ , that is, by the commutator of two half-twists intersecting transversally.

**Lemma 3.1.** Let  $x, y \in \tilde{B}_n$ .

- (i) If the endpoints of  $x$  and  $y$  are disjoint, then  $[x, y] = 1$ .
- (ii) If the endpoints of  $x$  and  $y$  have one common endpoint, the  $\langle x, y \rangle = 1$ .

*Proof.* [8, Section 3]. □

Let  $\tilde{\varphi}$  be the induced homomorphism from  $\varphi$ , s.t.  $\tilde{\varphi} : \tilde{B}_n \rightarrow S_n$ . Define  $\tilde{P}_n = \ker \tilde{\varphi}$ ,  $\tilde{P}_{n,0} = \ker \tilde{\varphi} \cap \ker \tilde{\delta}$  (where  $\tilde{\delta} : \tilde{B}_n \rightarrow \mathbb{Z}$ ).

We cite now the main results of [2, Section 1]; see also [12].

**Lemma 3.2.** Denote by  $x_i$  the image of the generator  $X_i$  in  $\tilde{B}_n$ . Let  $s_1 = x_1^2$ ,  $\mu = [x_1^2, x_2^2]$ ,  $u_i = [x_i^{-1}, x_{i+1}^2] \ \forall \ 1 \leq i \leq n-2$ ,  $u_{n-1} = [x_{n-2}^2, x_{n-1}^2]$ . So  $\tilde{P}_{n,0}$  is generated by  $u_1, \dots, u_{n-1}$ , and  $\tilde{P}_n$  is generated by  $s_1, u_1, \dots, u_{n-1}$ .

We also have the following:

$$[u_i, u_j] = \begin{cases} 1 & |i - j| > 1 \\ \mu & \text{otherwise} \end{cases}$$

$$[s_1, u_i] = \begin{cases} 1 & i \neq 2 \\ \mu & i = 2 \end{cases}$$

Moreover,  $\mu^2 = 1$ ,  $\mu \in \text{Center}(\tilde{B}_n)$  and  $\langle \mu \rangle = [\tilde{P}_{n,0}, \tilde{P}_{n,0}] = [\tilde{P}_n, \tilde{P}_n]$ . Therefore,  $\tilde{P}_{n,0}$  is solvable and  $\text{Ab}(\tilde{P}_n) \simeq \mathbb{Z}^n$ ,  $\text{Ab}(\tilde{P}_{n,0}) \simeq \mathbb{Z}^{n-1}$ .

We can also formulate the action of  $\tilde{B}_n$  on  $\tilde{P}_n$  by conjugation:

$$(s_1)_{x_i} = \begin{cases} s_1 & i \neq 2 \\ s_1 u_2^{-1} & i = 2 \end{cases}, \quad (u_j)_{x_i} = \begin{cases} u_j & |i - j| > 1 \\ u_i u_j & |i - j| = 1 \\ u_i^{-1} \mu & i = j \end{cases}$$

Actually, this action of  $\tilde{B}_n$  on  $\tilde{P}_n$  was developed (see [8]) to abstract groups with  $\tilde{B}_n$  actions similar to the action on  $\tilde{P}_n$  and  $P_{n,0}$ . This is explained in the following properties.

**Definition 3.4.** *Adjacent half-twists*

If  $x, y \in \tilde{B}_n$  are two half-twists whose endpoints have only one point in common (and they can intersect each other transversally), We say  $x$  and  $y$  are adjacent.

The following definitions, lemmas and propositions are taken from [12].

**Definition 3.5.** *Polarized half-twists, polarization*

We say that a half-twist  $X \in B_n$  (or  $\tilde{X}$  in  $\tilde{B}_n$ ) is polarized if we choose an order on the endpoints of  $X$ . The order is called the polarization of  $X$  or  $\tilde{X}$ .

**Definition 3.6.** *Orderly adjacent*

Let  $X, Y$  be two adjacent polarized half-twists in  $B_n$  (resp. in  $\tilde{B}_n$ ). We say that  $X, Y$  are *orderly adjacent* if their common point is the “end” of one of them and the “origin” of another.

The following definition derives its motivation from the action of  $\tilde{B}_n$  on  $\tilde{P}_n$ .

**Definition 3.7.**  *$\tilde{B}_n$ -group*

A group  $G$  is called a  $\tilde{B}_n$ -group if there exists a homomorphism  $\tilde{B}_n \rightarrow \text{Aut}(G)$ . We denote  $(g)_b$  by  $g_b$ .

**Definition 3.8.** *Prime element, supporting half-twist (s.h.t.) corresponding central element*

Let  $G$  be a  $\tilde{B}_n$ -group.

An element  $g \in G$  is called a *prime element* of  $G$  if there exists a half-twist  $X \in B_n$  and  $\tau \in \text{Center}(G)$  with  $\tau^2 = 1$  and  $\tau_b = \tau \forall b \in \tilde{B}_n$  such that

- (1)  $g_{\tilde{X}^{-1}} = g^{-1} \tau$
- (2) For every half-twist  $Y$  adjacent to  $X$  we have:
 
$$g_{\tilde{X}\tilde{Y}^{-1}\tilde{X}^{-1}} = g_{\tilde{X}}^{-1} g_{\tilde{X}\tilde{Y}^{-1}}$$

$$g_{\tilde{Y}^{-1}\tilde{X}^{-1}} = g^{-1} g_{\tilde{Y}^{-1}}.$$
- (3) For every half-twist  $Z$  disjoint from  $X$ ,  $g_{\tilde{Z}} = g$ .

The half-twist  $X$  (or  $\tilde{X}$ ) is called the *supporting half-twist* of  $g$  ( $X$  is the s.h.t. of  $g$ .)

The element  $\tau$  is called the *corresponding central element*.

**Lemma 3.1.** *Let  $G$  be a  $\tilde{B}_n$ -group.*

*Let  $g$  be a prime element in  $G$  with supporting half-twist  $X$  and corresponding central element  $\tau$ . Then:*

- (1)  $g_{\tilde{X}} = g_{\tilde{X}^{-1}} = g^{-1}\tau$ ,  $g_{\tilde{X}^2} = g$ .
- (2)  $g_{\tilde{Y}^{-2}} = g\tau \forall Y$  consecutive half-twist to  $X$ .
- (3)  $[g, g_{\tilde{Y}^{-1}}] = \tau \forall Y$  consecutive half-twist to  $X$ .

**Definition 3.9.** *Polarized pair*

Let  $G$  be a  $\tilde{B}_n$ -group,  $h$  a prime element of  $G$ ,  $X$  its supporting half-twist. If  $X$  is polarized, we say that  $(h, X)$  (or  $(h, \tilde{X})$ ) is a polarized pair with central element  $\tau$ ,  $\tau = hh_{\tilde{X}^{-1}}$ .

**Definition 3.10.** *Coherent pairs, anti-coherent pairs*

We say that two polarized pairs  $(h_1, \tilde{X}_1)$  and  $(h_2, \tilde{X}_2)$  are coherent (anti-coherent) if  $\exists \tilde{b} \in \tilde{B}_n$  such that  $(h_1)_{\tilde{b}} = h_2$ ,  $(\tilde{X}_1)_{\tilde{b}} = \tilde{X}_2$ , and  $\tilde{b}$  preserves (reverses) the polarization.

**Proposition 3.1.** *Let  $(h, \tilde{X})$  be a polarized pair,  $h \in G$ ,  $\tilde{X} \in \tilde{B}_n$ . Let  $\tilde{T}$  be a polarized half-twist in  $\tilde{B}_n$ . Then there exists a unique prime element  $g \in G$  such that  $(g, \tilde{T})$  and  $(h, \tilde{X})$  are coherent.*

**Definition 3.11.**  $L_{h, \tilde{X}}(\tilde{T})$

Let  $(h, X)$  be a polarized pair  $\tilde{T} \in \tilde{B}_n$ .  $L_{h, \tilde{X}}(\tilde{T})$  is the unique prime element s.t.  $(L_{h, \tilde{X}}(\tilde{T}), \tilde{T})$  is coherent with  $(h, \tilde{X})$ .

In fact, one can prove that  $\tilde{P}_n$  (as a  $\tilde{B}_n$ -group) has a prime element, and  $\tilde{P}_{n,0}$  is generated by the orbit of this prime element.

**Lemma 3.2.** *Let  $X_1, X_2$  be 2 consecutive half-twists in  $B_n$ . Let  $u = (\tilde{X}_1^2)_{\tilde{X}_2^{-1}} \tilde{X}_2^{-2}$ . Then  $u \in \tilde{P}_{n,0}$ ,  $u$  is a prime element in  $\tilde{P}_n$  (considered as a  $\tilde{B}_n$ -group), and  $\tilde{X}_1$  is the supporting half-twist of  $u$ .*

**Lemma 3.3.**  *$\tilde{P}_{n,0}$  is a primitive  $\tilde{B}_n$ -group generated by the  $\tilde{B}_n$ -orbit of a prime element  $u = \tilde{X}^2 \tilde{Y}^{-2}$ , where  $\tilde{X}, \tilde{Y}$  are adjacent half-twists in  $\tilde{B}_n$ ,  $\tilde{T} = \tilde{X} \tilde{Y} \tilde{X}^{-1}$  is a supporting half-twist for  $u$ .*

We shall also cite from [12] the criterion for prime elements in  $\tilde{B}_n$ -groups; we will not use it directly, but rather implicitly, when quoting, in Section 4, the results for the  $\tilde{B}_n$ -groups (see Lemma 4.2).

**Proposition 3.2.** *Assume  $n \geq 5$ . Let  $G$  be a  $\tilde{B}_n$ -group, and let*

$$(\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_{n-1})$$

*be a standard base of  $\tilde{B}_n$ . Let  $S$  be an element of  $G$  with the following properties:*

- (0)  $G$  is generated by  $\{S_b, b \in \tilde{B}_n\}$ ;
- (1<sub>a</sub>)  $S_{\tilde{X}_2^{-1} \tilde{X}_1^{-1}} = S^{-1} S_{\tilde{X}_2^{-1}}$ ;
- (1<sub>b</sub>)  $S_{\tilde{X}_1 \tilde{X}_2^{-1} \tilde{X}_1^{-1}} = S_{\tilde{X}_1}^{-1} S_{\tilde{X}_1, \tilde{X}_2^{-1}}$ ;
- (2) For  $\tau = SS_{\tilde{X}_1^{-1}}$ ,  $T = S_{\tilde{X}_2^{-1}}$ , we have:
  - (2<sub>a</sub>)  $\tau_{\tilde{X}_1^2} = \tau$ ;
  - (2<sub>b</sub>)  $\tau_T = \tau_{\tilde{X}_1}^{-1}$ ;
- (3)  $S_{\tilde{X}_j} = S \forall j \geq 3$ ;
- (4)  $S_c = S$ , where  $c = [\tilde{X}_1^2, \tilde{X}_2^2]$ .

*Then  $S$  is a prime element of  $G$ ,  $\tilde{X}_1$  is a supporting half-twist of  $S$  and  $\tau$  is the corresponding central element. In particular,  $\tau^2 = 1$ ,  $\tau \in \text{Center}(G)$ ,  $\tau_b = \tau \forall b \in \tilde{B}_n$ .*

## 4. CALCULATION OF THE FUNDAMENTAL GROUP

In this section we will calculate the fundamental group of the complement of the branch curve of  $F_{1,(a,b)}$ . This computation requires explicit knowledge of the braid monodromy factorization (BMF) technique. This knowledge can be found at [10],[6], [7]. However, we recall the main definitions regarding the braid monodromy factorization related to a curve  $S$ .

**Definition 4.1.** The braid monodromy w.r.t.  $S, \pi, u$

Let  $S$  be a curve,  $S \subseteq \mathbb{C}^2$ . Let  $\pi : S \rightarrow \mathbb{C}^1$  be defined by  $\pi(x, y) = x$ . We denote  $\deg \pi$  by  $m$ . Let  $N = \{x \in \mathbb{C}^1 \mid \#\pi^{-1}(x) < m\}$ . Take  $u \notin N$ ,  $u$  real, s.t.  $\Re(x) \ll u \quad \forall x \in N$ . Let  $\mathbb{C}_u^1 = \{(u, y)\}$ . There is a natural defined homomorphism

$$\pi_1(\mathbb{C}^1 - N, u) \xrightarrow{\varphi} B_m[\mathbb{C}_u^1, \mathbb{C}_u^1 \cap S]$$

which is called *the braid monodromy w.r.t.  $S, \pi, u$* , where  $B_m$  is the braid group. We sometimes denote  $\varphi$  by  $\varphi_u$ . Note that in this definition we regard  $B_m$  as the group of diffeomorphisms, as described in the previous section.

Denote the generator of the center of  $B_n$  as  $\Delta^2$ . We recall Artin's theorem on the presentation of  $\Delta^2$  as a product of braid monodromy elements of a geometric-base (a base of  $\pi_1 = \pi_1(\mathbb{C}^1 - N, u)$  with certain properties; see [4] for definitions).

**Theorem:** Let  $S$  be a curve transversal to the line in infinity, and  $\varphi$  is a braid monodromy of  $S, \varphi : \pi_1 \rightarrow B_m$ . Let  $\delta_i$  be a geometric (free) base (g-base) of  $\pi_1$ . Then:

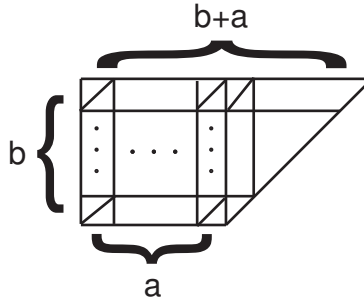
$$\Delta^2 = \prod \varphi(\delta_i).$$

This product is also defined as the *braid monodromy factorization* (BMF) related to a curve  $S$ .

Since  $\tilde{S}_{F_{1,(a,b)}}$ , which is the branch curve of the degenerated surface  $\tilde{F}_{1,(a,b)}$ , is a line arrangement, we can compute the braid monodromy factorization as in [4]. In order to compute the braid monodromy factorization of  $S_{F_{1,(a,b)}}$ , we use the regeneration rules ([7]). The regeneration methods are actually, locally, the reverse process of the degeneration method. When regenerating a singular configuration consisting of lines and conics, the final stage in the regeneration process involves doubling each line, so that each point of  $K$  (which is the set of points in the disk, that is  $\mathbb{C}_u^1 \cap \tilde{S}_{F_{1,(a,b)}}$ ) corresponding to a line labelled  $i$  is replaced by a pair of points, labelled  $i$  and  $i'$ . The purpose of the regeneration rules is to explain how the braid monodromy behaves when lines are doubled in this manner.

Let  $F_{1,(a,b)}$ ,  $a, b > 1$  be the Hirzebruch surface embedded w.r.t. the linear system  $|aC + bE_0|$ . As shown,  $F_{1,(a,b)}$  could be degenerated into a union of  $2ab + b^2$  planes in the following arrangement:





(figure 4.1)

We shall give a special presentation of  $B_n$ , from which we will induce an injection of  $\tilde{B}_n$  to  $G = \pi_1(\mathbb{C}^2 - S_{F_{1,(a,b)}})$ .

**Remark:** From now on, we denote by  $\tilde{S}_{F_{1,(a,b)}}$  the branch curve of  $F_{1,(a,b)}$  (in  $\mathbb{CP}^2$ ), and by  $S_{F_{1,(a,b)}}$  a generic affine portion of it (in  $\mathbb{C}^2$ ).

Let  $a, b$  be integers  $b > 1, a \geq 1, n = 2ab + b^2$ . Let  $s_{ij} = (i, j), t_{ij} = (i + \frac{1}{2}, j) \in \mathbb{R}^2$ . Let  $K_{a,b}$  be the set in  $\mathbb{R}^2$  consisting of the points  $s_{ij}, t_{ij}, 1 \leq j \leq b, 1 \leq i \leq a + j$ ; so  $\#K_{a,b} = 2ab + b^2$ .

Let  $D$  be a large disk in  $\mathbb{R}^2$  containing  $K_{a,b}$ . Consider the oriented line segments  $\vec{x}_{ij} = [s_{i,j}, t_{i,j}]$  where  $1 \leq j \leq b, 1 \leq i \leq a + j$ ;  $\vec{y}_{ij} = [t_{i,j}, s_{i+1,j}]$ ,  $1 \leq j \leq b, 1 \leq i \leq a + j - 1$ ;  $\vec{z}_{ij} = [s_{i,j}, t_{i,j+1}]$ ,  $1 \leq j \leq b - 1, 1 \leq i \leq a + j$ . Consider  $B_n = B_n[D, K_{a,b}]$ . Let  $X_{ij}, Y_{ij}, \underline{Z}_{ij}$  be polarized half-twists in  $B_n$  by the oriented segments  $\vec{x}_{ij}, \vec{y}_{ij}, \vec{z}_{ij}$  respectively. Let  $Z_{ij} = \underline{Z}_{ij}$ , when  $i = a + j, 1 \leq j \leq b - 1$ . We define  $Z_{ij}$  for  $1 \leq i \leq a + j - 1$ , inductively:

$$Z_{ij} = (Z_{i+1,j})_{X_{i+1,j+1}^{-1} Y_{i,j+1} Y_{i,j}^{-1} X_{i,j}}.$$

**Proposition 4.1.**  $B_n$  can be finitely presented as follows:

*Generators:*

$$\begin{aligned} X_{i,j}, & 1 \leq j \leq b, 1 \leq i \leq a + j. \\ Y_{ij}, & 1 \leq j \leq b; 1 \leq i \leq a + j - 1. \\ Z_{ij}, & i = a + j, 1 \leq j \leq b - 1. \end{aligned}$$

*Relations:*

$$\begin{aligned} & \forall \text{ two generators } a, b \text{ of the above which are adjacent, } \langle a, b \rangle = 1. \\ & \forall \text{ two generators } c, d \text{ which are disjoint } [c, d] = 1. \\ & \forall j \in (1, \dots, b - 1), i = a + j : [X_{i,j}, Z_{ij} Y_{i-1,j} Z_{ij}^{-1}] = 1. \end{aligned}$$

*Proof.* This is a standard consequence of the usual presentation of  $B_n[D, K_{a,b}]$  (see [4]).  $\square$

The formulas define inductively a polarization for each  $Z_{ij}$ . One can check that it coincides with the given polarization of  $\underline{Z}_{ij}$ , i.e., corresponds to the ordered pair  $(s_{ij}, t_{i,j+1})$ .

Denote by  $x_{ij}, y_{ij}, z_{ij}$  the images of  $X_{ij}, Y_{ij}, Z_{ij}$  in  $\tilde{B}_n$ . Thus we get a representation of  $\tilde{B}_n$ . We consider  $\{x_{ij}, y_{ij}, z_{ij}\}$  with polarization introduced above.

**Definition 4.2.** Let  $G$  be a primitive  $\tilde{B}_n$ -group generated by the orbit of a prime element  $B_{1,1}$  supported by the half-twist  $Y_{1,1}$ . According to Proposition 3.1,  $\forall$  polarized half-twist  $t \in \tilde{B}_n, \exists$  unique prime element  $L_{\{B_{1,1}, y_{1,1}\}}(t) \in G$ , s.t. the pair  $\{L_{\{B_{1,1}, y_{1,1}\}}(t), t\}$  is coherent with  $\{B_{1,1}, y_{1,1}\}$ .

Define

$$\begin{aligned} A_{ij} &= L_{\{B_{1,1}, y_{1,1}\}}(x_{ij}) \\ B_{ij} &= L_{\{B_{1,1}, y_{1,1}\}}(y_{ij}) \\ C_{ij} &= L_{\{B_{1,1}, y_{1,1}\}}(z_{ij}) \end{aligned}$$

**Remark 4.1.** Looking at [2, Remark 6], one gets the formulas for the  $\tilde{B}_n$ -action on  $G$  in terms of  $\{x_{ij}, y_{ij}, z_{ij}; A_{ij}, B_{ij}, C_{ij}; i, j = \dots\}$ . In particular, we see that  $G$  is generated by  $\{A_{ij}, B_{ij}, C_{ij}\}$  (because  $G$  is generated by the  $\tilde{B}_n$ -orbit of  $B_{1,1}$ ).

Denote by  $\tilde{S}_{a,b} := \tilde{S}_{F_{1,(a,b)}}$  the degenerated branch curve of  $F_{1,(a,b)}$ . We define now a planar 2-complex, to represent the polygon in Fig. 4.1.

**Definition 4.3.** We use a planar 2-complex  $K(a, b)$  defined as follows:  $K(a, b) \subset \mathbb{R}^2$ . Define  $P$ , the polygon whose vertices are  $(0, 0), (a, 0), (a + b, b), (0, b)$ . So the vertices of  $K(a, b)$  are the points  $\left\{ \omega_{rk} = (r, k) \mid \begin{smallmatrix} \omega_{rk} \in P \\ r, k \in \mathbb{Z} \end{smallmatrix} \right\}$ . The edges of  $K(a, b)$  are the straight line segments of the following three types:

- (a) “diagonal”:  $[\omega_{r,k}, \omega_{r+1,k+1}], 0 \leq k \leq b - 1, 0 \leq r \leq a + k;$
- (b) “vertical”:  $[\omega_{r,k}, \omega_{r,k+1}], 0 \leq k \leq b - 1, 0 \leq r \leq a + k;$
- (c) “horizontal”:  $[\omega_{r,k}, \omega_{r+1,k}], 0 \leq k \leq b, 0 \leq r \leq a + k - 1;$

The 2-simplices of  $K(a, b)$  are the triangles  $\Delta\{\omega_{r,k}, \omega_{r+1,k}, \omega_{r+1,k+1}\}$  and  $\Delta\{\omega_{r,k}, \omega_{r,k+1}, \omega_{r+1,k+1}\}$ .

**Definition 4.4.** The vertices  $\omega_{rk}$  that are not on the boundary of  $P$  will be called 6-point; the vertices  $\omega_{0,0}, \omega_{a,0}$  will be called 2-point; and all the other vertices  $\omega_{rk}$  on the boundary of  $P$  s.t.  $(r, k) \neq (0, b), (a + b, b)$  will be called 3-point.

**Definition 4.5. (1)** Consider  $B_m = B_m[D, K]$ , where  $D$  is a large disk in  $\mathbb{C}^1$ , centered at  $(0)$  and

$$\begin{aligned} K &= \{q_{rk\delta}^{(\varepsilon)} \mid \varepsilon = 1, 2, 3, \delta = 0, 1 \text{ s.t. :} \\ &\quad \text{for } \varepsilon = 1, 1 \leq k \leq b, 1 \leq r \leq a + k - 1 \\ &\quad \text{for } \varepsilon = 2, 1 \leq k \leq b, 1 \leq r \leq a + k - 1 \\ &\quad \text{for } \varepsilon = 3, 1 \leq k \leq b - 1, 1 \leq r \leq a + k \\ &\quad q_{rk\delta}^{(\varepsilon)} \text{ are real points such that } q_{rk0}^{(\varepsilon)}, q_{rk1}^{(\varepsilon)} \text{ are very close to each other, and} \\ &\quad q_{rk\delta}^{(\varepsilon)} < q_{r'k'\delta'}^{(\varepsilon')} \text{ if either } k < k' \text{ or } k = k' \text{ and } r < r' \\ &\quad \text{or } k = k', r = r' \text{ and } \varepsilon < \varepsilon', \text{ or } k = k', r = r', \varepsilon = \varepsilon' \text{ and } \delta < \delta'\}. \end{aligned}$$

The points of  $K$  that we associate with the non-boundary edges of  $K(a, b)$  are as follows:  $q_{rk0}^{(1)}, q_{rk1}^{(1)}$  correspond to the diagonal edge  $[\omega_{r-1,k-1}, \omega_{r,k}]$ ;  $q_{rk0}^{(2)}, q_{rk1}^{(2)}$  correspond to the vertical edge  $[\omega_{r,k-1}, \omega_{r,k}]$ ;  $q_{rk0}^{(3)}, q_{rk1}^{(3)}$  correspond to the horizontal edge  $[\omega_{r-1,k}, \omega_{r,k}]$ .

As was indicated earlier, during the regeneration process, each line doubles itself, and thus each point of  $\mathbb{C}^1 \cap \tilde{S}_{a,b}$  is replaced by a pair of points, which are  $q_{rk0}^{(\varepsilon)}$  and  $q_{rk1}^{(\varepsilon)}$ .

(2) Let

$$m_{r,k} = \begin{cases} 12 & \text{if } \omega_{rk} \text{ is a 6-point} \\ 4 & \text{if } \omega_{rk} \text{ is a 3-point} \\ 2 & \text{if } \omega_{rk} \text{ is a 2-point} \end{cases}$$

Denote by  $K_{r,k}$  the subset of  $K$  consisting of the points associated with the non-boundary edges of  $K(a, b)$  which meet at  $\omega_{r,k}$ . Clearly,  $\#K_{r,k} = m_{r,k}$ .

(3) Denote  $f_{rk} : B_{m_{r,k}} \rightarrow B_m[D, K]$  an embedding of  $B_{m_{r,k}}$  into  $B_m[D, K]$  corresponding to a connection below the real axis of the points of  $K_{r,k}$  by consecutive simple paths (see [4]). Clearly, each  $B_{m_{r,k}}$  is either  $B_{12}$ ,  $B_4$  or  $B_2$ .

From each 6/3/2-point, relations between the generators of the fundamental group  $\pi_1(\mathbb{C}^2 - S_{F_{1,(a,b)}})$  can be induced. These relations are written with the same notations as in [2]. We refer the reader to this article. However, we state a few of the main results.

Consider  $K \subset D$ ,  $K = \{q_{rk\delta}^{(\varepsilon)}\}$ . Take a point  $\underline{u}$  on  $\partial D$  below the real axis. Using small (positively oriented) circles around the points  $q_{rk\delta}^{(\varepsilon)}$  and connecting these circles by (straight) simple lines with  $\underline{u}$ , we obtain a geometric base  $\{\gamma_{rk\delta}^{(\varepsilon)}\}$  for  $\pi_1(D - K, \underline{u})$ .

A full set of relations between  $\{\gamma_{rk\delta}^{(\varepsilon)}\}$  can be described, corresponding to the braid monodromy factorization (see [4] for a formula computing the BMF of a generic line arrangement - which is actually the factorization on which we perform the regeneration process to get the following):

$$\Delta^2 = \varepsilon(a, b) = \prod_{\omega_{r,k}} C(r, k) \mathcal{H}(r, k),$$

where  $\mathcal{H}(r, k)$  are the factorizations induced from the 6/3/2-points -  $\omega_{r,k}$  (see Appendix).  $C(r, k)$  are the factorizations that we get from the parasitic intersection of the branch curves (see [2, Chapter 2] or [4]). We get a presentation of  $\pi_1(\mathbb{C}^2 - S_{F_{1,(a,b)}})$  by using the Van-Kampen Theorem [16] which says that from each factor from  $\varepsilon(a, b)$ , a relation between  $\{\gamma_{rk\delta}^{(\varepsilon)}\}$  can be induced. Taking a braid which is a half-twist that corresponds to a path  $\sigma$  from  $q_{r_1 k_1 \delta_1}^{\varepsilon_1}$  to  $q_{r_2 k_2 \delta_2}^{\varepsilon_2}$  via  $u$ , we let  $\delta_1$  (resp.  $\delta_2$ ) be the path from  $u$  to  $q_{r_1 k_1 \delta_1}^{\varepsilon_1}$  (resp.  $q_{r_2 k_2 \delta_2}^{\varepsilon_2}$ ) along  $\sigma$ , going around  $q_{r_1 k_1 \delta_1}^{\varepsilon_1}$  (resp.  $q_{r_2 k_2 \delta_2}^{\varepsilon_2}$ ) and coming back to  $u$  along the same path, respectively. Let  $A$  and  $B$  be the homotopy classes of a loop around  $q_{r_1 k_1 \delta_1}^{\varepsilon_1}$  (resp.  $q_{r_2 k_2 \delta_2}^{\varepsilon_2}$ ) along  $\delta_1$  (resp.  $\delta_2$ ).  $A$  (resp.  $B$ ) is a conjugation of  $\gamma_{r_1 k_1 \delta_1}^{\varepsilon_1}$  (resp.  $\gamma_{r_2 k_2 \delta_2}^{\varepsilon_2}$ ).

By the Van Kampen Theorem, we have one of the following relations in  $\pi_1(\mathbb{C}^2 - S_{F_{1,(a,b)}})$  (fixed according to the type of singularity, from which we have the path  $\sigma$ ):

1.  $A = B$ , if the singularity is a branch point,
2.  $[A, B] = ABA^{-1}B^{-1} = e$  if it is a node,
3.  $\langle A, B \rangle = ABAB^{-1}A^{-1}B^{-1} = e$  if it is a cusp.

**Definition 4.6.** Let

$$\begin{aligned} \ell_{r,k}^{(1)} &= \begin{cases} 1 - k & \text{for } r \geq k \\ 1 - r & \text{for } r < k \end{cases} \\ \ell_{r,k}^{(2)} &= k - 1 \\ \ell_{r,k}^{(3)} &= 0. \end{aligned}$$

(Evidently,  $\ell_{r+1,k}^{(3)} = \ell_{r,k}^{(3)}$ ;  $\ell_{r,k+1}^{(2)} = \ell_{r,k}^{(2)} + 1$ ;  $\ell_{r+1,k+1}^{(1)} = \ell_{r,k}^{(1)} - 1$ .) Let  $e_{rk\delta}^{(\varepsilon)} = (\gamma_{rk\delta}^{(\varepsilon)})(\rho_{rk}^{(\varepsilon)})^{\ell_{rk}^{(\varepsilon)}}$  (where  $\rho_{rk}^{(\varepsilon)}$  is the half-twist in  $B_m[D, K]$  defined by the segment  $[q_{rk0}, q_{rk1}]$ ).

**Definition 4.7.** Denote by  $G$  the group defined by  $\varepsilon(a, b)$ ; more precisely, the quotient of the free group generated by  $\{e_{rk\delta}^{(\varepsilon)}\}$ , modulo relations (we call them  $R\varepsilon$ ) induced from

6/3/2-points, and the relation induces from the parasitic intersections, for all  $\omega_{r,k}$  (see [2, Chapter 3] for those relations or in the Appendix).

By the definition of  $\varepsilon(a, b)$  (braid monodromy factorization for  $S_{F_{1,(a,b)}}$ ), we have  $G \simeq \pi_1(\mathbb{C}^2 - S_{F_{1,(a,b)}}, \underline{u})$ . Let  $E_{rk\delta}^{(\varepsilon)}$  be the images of  $e_{rk\delta}^{(\varepsilon)}$  in  $G$ .

**Proposition 4.2.**  $\exists$  homomorphism  $\tilde{\alpha} : \tilde{B}_n \rightarrow G$  which is defined by:

$$\begin{aligned}\tilde{\alpha}(x_{ij}) &= E_{ij0}^{(1)}, & \tilde{\alpha}(y_{ij}) &= E_{ij0}^{(2)} \quad \forall i, j, \\ \tilde{\alpha}(z_{ij}) &= E_{ij0}^{(3)} \quad (\text{where } i = a + j);\end{aligned}$$

moreover,

$$\tilde{\alpha}(z_{ij}) = E_{ij0}^{(3)} \quad \forall (i, j) \in \text{Vertices}(K(a, b)), \quad i \neq a + j.$$

*Proof.* See [2, Proposition 8]. See the induced relations for each 2/3/6-point and explanation why  $\tilde{B}_n$  can be embedded in  $G$  in the Appendix.  $\square$

Let  $E_{rk}^{(\varepsilon)} = E_{rk0}^{(\varepsilon)}$ ,  $\mathcal{B}$  be the subgroup of  $G$  generated by  $\{E_{rk}^{(\varepsilon)}\}$ . It follows from Proposition 4.2 that  $\mathcal{B} = \tilde{\alpha}(\tilde{B}_n)$ . Let  $\mathcal{P} = \tilde{\alpha}(\tilde{P}_n)$ ,  $\mathcal{P}_0 = \tilde{\alpha}(\tilde{P}_{n,0})$  (where  $P_{n,0} = \ker(P_n \rightarrow \text{Ab}(B_n))$ ,  $\tilde{P}_{n,0}$  is the image of  $P_{n,0}$  in  $\tilde{B}_n$ ). From [2, Theorem 1] or from Lemma 3.3, it follows that  $\tilde{P}_{n,0}$  is a primitive  $\tilde{B}_n$ -group with prime element  $u = (y_{1,1}^2)_{x_{2,1}^{-1}x_{2,1}^{-2}}$  ( $x_{2,1}$  and  $(y_{1,1})_{x_{2,1}^{-1}}$  are two adjacent half-twists in  $\tilde{B}_n$ ), and s.h.t. equal to  $y_{11}$ . Denote  $c = [y_{1,1}^2, x_{2,1}^2]$ . We get from [2, Theorem 1] that  $c^2 = 1$ ,  $c \in \text{Center}(\tilde{B}_n)$  and  $c$  generates  $\tilde{P}'_n$  and  $\tilde{P}'_{n,0}$ . Denoting  $\eta_{1,1} = \tilde{\alpha}(u) = (E_{1,1}^{(2)})^2_{(E_{2,1}^{(1)})^{-1}} \cdot (E_{2,1}^{(1)})^{-2}$ ,  $\mu = \tilde{\alpha}(c) = [(E_{1,1}^{(2)})^2, (E_{2,1}^{(1)})^2]$ , we get that  $\mathcal{P}_0$  is a primitive  $\tilde{B}_n$ -group,  $\eta_{1,1}$  is a prime element of  $\mathcal{P}_0$  with s.h.t.  $y_{1,1}$ ,  $\mu^2 = 1$ ,  $\mu \in \text{Center}(\mathcal{B})$  and  $\mu$  generates  $\mathcal{P}'$  and  $\mathcal{P}'_0$ . Using the polarization of  $X_{i,j}Y_{i,j}$ ,  $Z_{i,j}$  and Proposition 4.1, we can find  $\forall t \in \{x_{ij}, y_{ij}, z_{ij}\}$  (the generators of  $\tilde{B}_n$ ) and  $\{z_{i,j} \mid (i, j) \in \text{Vertices}(K(a, b)), i, j \geq 1, i \neq a + j\}$  unique  $L_{\{\eta_{1,1}, y_{1,1}\}}(t) \in \mathcal{P}_0$  s.t. the pair  $\{L_{\{\eta_{1,1}, y_{1,1}\}}(t), t\}$  is coherent with  $\{\eta_{1,1}, y_{1,1}\}$ .

**Definition 4.8.** Recall that  $u = (y_{11}^2)_{x_{2,1}^{-1}x_{2,1}^{-2}}$ ,  $\eta_{1,1} = \tilde{\alpha}(u)$ . Define

$$\begin{aligned}\xi_{i,j} &= L_{\{\eta_{1,1}, y_{1,1}\}}(x_{ij}), & \eta_{i,j} &= L_{\{\eta_{1,1}, y_{1,1}\}}(y_{i,j}) \\ \zeta_{i,j} &= L_{\{\eta_{1,1}, y_{1,1}\}}(z_{i,j}).\end{aligned}$$

**Lemma 4.1.**  $\mu \in \text{Center}(G)$ .

*Proof.* See [2, Lemma 16].  $\square$

**Definition 4.9.** Let

$$d_{rk} = E_{rk1}^{(1)}(E_{rk0}^{(1)})^{-1}, \quad v_{rk} = E_{rk1}^{(2)}(E_{rk0}^{(2)})^{-1}, \quad h_{rk} = E_{rk1}^{(3)}(E_{rk0}^{(3)})^{-1}.$$

( $d, v, h$  correspond to “diagonal”, “vertical”, “horizontal”.) Clearly,  $G$  is generated by  $\{d_{rk}, v_{rk}, h_{rk}; r, k = \dots\}$  and  $\mathcal{B}$ . Denote by  $\mathcal{H}$  the subgroup of  $G$  generated by the  $\mathcal{B}$ - (or  $\tilde{B}_n$ -) orbit of  $v_{1,1}$ .

**Lemma 4.2.**

- 1)  $\mathcal{H}$  is a primitive  $\tilde{B}_n$ -group with prime element  $v_{1,1}$ , s.h.t.  $y_{1,1}$ .
- 2)  $v_{1,1}$  is actually a prime element of  $G$  with s.h.t.  $y_{1,1}$  (i.e.,  $v_{1,1} \cdot (v_{1,1})_{y_{1,1}^{-1}} \in \text{Center}(G)$ ).

*Proof.* As in [2, Lemma 17].  $\square$

**Definition 4.10.** Using the polarization of  $X_{i,j}, Y_{i,j}, Z_{i,j}$ , we find  $\forall t \in \{x_{ij}, y_{ij}, z_{ij}\} \exists ! L_{\{v_{1,1}, y_{1,1}\}}(t) \in \mathcal{H}$  s.t. the pair  $\{L_{\{v_{1,1}, y_{1,1}\}}(t), t\}$  is coherent with  $\{v_{1,1}, y_{1,1}\}$ . Define

$$a_{i,j} = L_{\{v_{1,1}, y_{1,1}\}}(x_{ij}), \quad b_{i,j} = L_{\{v_{1,1}, y_{1,1}\}}(y_{ij}), \quad c_{i,j} = L_{\{v_{1,1}, y_{1,1}\}}(z_{ij}).$$

**Remark 4.2.**  $\xi_{ij}, \eta_{ij}, \zeta_{ij}$  ( $a_{ij}, b_{ij}, c_{ij}$ ) coincide with  $A_{ij}, B_{ij}, C_{ij}$  introduced in Definition 4.2 for an arbitrary primitive  $\tilde{B}_n$ -group  $G$ , when this  $G$  is replaced by  $\mathcal{P}_0$  (resp.  $\mathcal{H}$ ), and  $\{B_{11}, Y_{11}\}$  is replaced by  $\{\eta_{11}, y_{11}\}$  (resp.  $\{v_{11}, y_{11}\}$ ). Therefore, replacing  $A_{ij}, B_{ij}, C_{ij}$  by  $\xi_{ij}, \eta_{ij}, \zeta_{ij}$  (resp.  $a_{ij}, b_{ij}, c_{ij}$ ), we obtain formulas expressing the  $\tilde{B}_n$ -action on  $\mathcal{P}_0$  (resp. on  $\mathcal{H}$ ). In particular,  $\mathcal{P}_0$  (resp.  $\mathcal{H}$ ) is generated by  $\{\xi_{ij}, \eta_{ij}, \zeta_{ij}\}$  (resp.  $\{a_{ij}, b_{ij}, c_{ij}\}$ ).

**Definition 4.11.**  $\forall x_{i,j}, y_{i,j}, z_{i,j}$ , let  $\tilde{x}_{i,j} = \tilde{\alpha}(x_{i,j}), \tilde{y}_{i,j} = \tilde{\alpha}(y_{i,j}), \tilde{z}_{i,j} = \tilde{\alpha}(z_{i,j})$ .

**Remark 4.3.** We have, by [2, Remark 30], the following:

$$\begin{aligned} d_{r+1,k+1} &= (d_{rk})_{\tilde{z}_{rk} \tilde{y}_{rk} \tilde{z}_{r+1,k}^{-1} \tilde{y}_{r,k+1}^{-1}} \\ h_{r+1,k} &= (h_{rk})_{\tilde{x}_{rk}^{-1} \tilde{y}_{rk} \tilde{y}_{r,k+1}^{-1} \tilde{x}_{r+1,k+1}} \\ v_{r,k+1} &= (v_{rk})_{\tilde{x}_{rk}^{-1} \tilde{z}_{rk} \tilde{z}_{r+1,k}^{-1} \tilde{x}_{r+1,k+1}} \\ h_{rk} &= (v_{rk} d_{rk} (v_{rk}^{-1})_{x_{rk}^{-1}})_{z_{rk} x_{rk}} \\ v_{rk} &= (h_{rk} d_{rk} (h_{rk}^{-1})_{x_{rk}^{-1}})_{y_{rk} x_{rk}}. \end{aligned}$$

**Remark 4.4.** By [2, Remark 31], we have

$$\begin{aligned} d_{r+1,1} &= \tilde{y}_{r1}^{-2} (v_{r,1})_{x_{r+1,1}^{-1} y_{r,1}^{-1}} \cdot (\tilde{y}_{r,1}^2)_{x_{r+1,1}^{-1}} \\ d_{1,k+1} &= \tilde{z}_{1,k}^{-2} (h_{1,k})_{x_{1,k}^{-1} z_{1,k}^{-1}} \cdot (\tilde{z}_{1,k}^2)_{x_{1,k+1}^{-1}} \\ v_{r,b} &= \tilde{x}_{r,k}^{-2} \cdot (d_{r,b})_{y_{r,b}^{-1} x_{r,b}^{-1}} \cdot (\tilde{x}_{r,b}^2)_{y_{r,b}^{-1}} \end{aligned}$$

Notice that in the following calculation, we will use the fact that  $\mu^2 = \nu^2 = 1$  (since they are central elements).

**Proposition 4.3.** Let  $\lambda(k) = \frac{k(k-1)}{2}$ . We have

$$\begin{aligned} h_{rk} &= c_{rk}^k \zeta_{rk}^{-k+1} (\mu\nu)^{\lambda(k)} & \forall r, k \\ d_{rk} &= a_{rk}^{r-k} \xi_{rk}^{-r+k} (\mu\nu)^{\lambda(k-r)} & \forall r, k \\ v_{rk} &= b_{rk}^r \eta_{rk}^{-r+1} (\mu\nu)^{\lambda(r)} & \forall r < a \end{aligned}$$

*Proof.* See [2, Proposition 10]. □

**Proposition 4.4.**  $v_{a,k} = 1, \forall 0 \leq k \leq b$ .

*Proof.* By the definition,  $v_{a,0} = E_{a01}^{(2)} (E_{a00}^{(2)})^{-1}$ , but  $\omega_{a,0}$  is a 2-point, and the induced relation from it is  $\gamma_{a00}^{(2)} = \gamma_{a01}^{(2)}$  or  $1 = E_{a01}^{(2)} (E_{a00}^{(2)})^{-1}$ . by the relation  $v_{r,k+1} = (v_{r,k})_{\tilde{x}_{rk}^{-1} \tilde{z}_{rk} \tilde{z}_{r+1,k}^{-1} \tilde{x}_{r+1,k+1}}$ , we can see that  $v_{a,k} = 1 \forall 0 \leq k \leq b$ . □

**Proposition 4.5.** For  $(r, k) \in \{(a+1, 2), (a+2, 3), \dots, (a+b-1, b-1)\} =: I$ ,

$$v_{r,k} = (E_{r,k-1}^{(3)})^{-2} h_{r,k-1}^{-1} (h_{r,k-1})_{(E_{r,k}^{(2)})^{-1}} (E_{r,k-1}^{(3)})_{(E_{r,k}^{(2)})^{-1}}^2.$$

*Proof.* Assume  $(r, k) = (a + 1, 2)$ . The proof for the other points is the same.

We have by the relations induced from the 3-point  $\omega_{a+1,2}$  :

$$E_{a+1,2,1}^{(2)} = (E_{a+1,2,0}^{(2)})_{(E_{a+1,1,1}^{(3)})^{-1}(E_{a+1,1,0}^{(3)})^{-1}},$$

or

$$\begin{aligned} v_{a+1,2} E_{a+1,2,0}^{(2)} &= (E_{a+1,1,0}^{(3)})^{-2} h_{a+1,1}^{-1} E_{a+1,2,0}^{(2)} h_{a+1,1} (E_{a+1,1,0}^{(3)})^2 \\ v_{a+1,2} &= (E_{a+1,1,0}^{(3)})^{-2} h_{a+1,1}^{-1} (h_{a+1,1})_{(E_{a+1,2,0}^{(2)})^{-1}} (E_{a+1,1,0}^{(3)})^2_{(E_{a+1,2,0}^{(2)})^{-1}}. \end{aligned}$$

By abuse of notation, we remove the last index from the  $E_{\dots}$ . □

We know that  $\eta_{rk}$  (for  $(r, k) \in I$ ) is a prime element with s.h.t.  $y_{rk}$  and a central element  $\mu$ . So it can be proven (see [14, Claim 5.5]) that

$$\eta_{r,k} = (E_{r,k-1}^{(3)})^2 (E_{r,k-1}^{(3)})_{(E_{r,k}^{(2)})^{-1}}^{-2}$$

or

$$\mu \eta_{r,k}^{-1} = (E_{r,k-1}^{(3)})^{-2} (E_{r,k-1}^{(3)})_{(E_{r,k}^{(2)})^{-1}}^2. \quad (4.1)$$

So we have

$$\begin{aligned}
v_{r,k} &= (E_{r,k-1}^{(3)})^{-2} h_{r,k-1}^{-1} (h_{r,k-1})_{(E_{r,k}^{(2)})^{-1}} (E_{r,k-1}^{(3)})^2_{(E_{r,k}^{(2)})^{-1}} \\
&\stackrel{[8, \text{IV.6.1}]}{=} h_{r,k-1}^{-1} (E_{r,k-1}^{(3)})^{-2} (E_{r,k-1}^{(3)})_{(E_{r,k}^{(2)})^{-1}} (h_{r,k-1})_{(E_{r,k}^{(2)})^{-1}} \\
&= h_{r,k-1}^{-1} \mu \eta_{r,k}^{-1} (h_{r,k-1})_{(E_{r,k}^{(2)})^{-1}}.
\end{aligned} \tag{4.2}$$

We compute now  $(h_{r,k-1})_{(E_{r,k}^{(2)})^{-1}}$ . We know that  $(c_{r,k-1})_{(E_{r,k}^{(2)})^{-1}} = c_{r,k-1} b_{r,k}$  ([8, IV.6.3]) and  $(\zeta_{r,k-1})_{(E_{r,k}^{(2)})^{-1}} = \zeta_{r,k-1} \eta_{r,k}$ . So

$$\begin{aligned}
(h_{r,k-1})_{(E_{r,k}^{(2)})^{-1}} &\stackrel{\text{Proposition 4.3}}{=} (\mu\nu)^{\lambda(k-1)} (c_{r,k-1}^{k-1} \zeta_{r,k-1}^{-k+2})_{(E_{r,k}^{(2)})^{-1}} \\
&= (\mu\nu)^{\lambda(k-1)} (c_{r,k-1})_{(E_{r,k}^{(2)})^{-1}}^{k-1} (\zeta_{r,k-1})_{(E_{r,k}^{(2)})^{-1}}^{-k+2} \\
&= (\mu\nu)^{\lambda(k-1)} (c_{r,k-1} b_{r,k})^{k-1} (\zeta_{r,k-1} \eta_{r,k})^{-k+2} \\
&= (\mu\nu)^{\lambda(k-1)} \nu^{\lambda(k-1)} c_{r,k-1}^{k-1} b_{r,k}^{k-1} \mu^{\lambda(k-2)} \eta_{r,k}^{-k+2} \zeta_{r,k-1}^{-k+2}.
\end{aligned} \tag{4.3}$$

We substitute the expressions we found in 4.3, (4.1), (4.3) in (4.2), and we get (for  $(r, k) \in I$ ):

$$\begin{aligned}
v_{r,k} &= (\mu\nu)^{\lambda(k-1)} \zeta_{r,k-1}^{-k+2} c_{r,k-1}^{1-k} \cdot \mu \nu_{r,k}^{-1} (\mu\nu)^{\lambda(k-1)} \nu^{\lambda(k-1)} c_{r,k-1}^{k-1} b_{r,k}^{k-1} \\
&\quad \cdot \mu^{\lambda(k-2)} \eta_{r,k}^{-k+2} \zeta_{r,k-1}^{-k+2} \\
\left( \begin{array}{c} \mu^2 = 1 \\ \nu^2 = 1 \end{array} \right) &= \mu^{\lambda(k-2)+1} \nu^{\lambda(k-1)} \zeta_{r,k-1}^{-k+2} c_{r,k-1}^{1-k} \eta_{r,k}^{-1} c_{r,k-1}^{k-1} b_{r,k}^{k-1} \eta_{r,k}^{-k+2} \zeta_{r,k-1}^{-k+2} \\
\left( \begin{array}{c} [c_{r,k-1}, \eta_{r,k}^{-1}] \\ = \nu \end{array} \right) &= \nu^{k-1+\lambda(k-1)} \mu^{\lambda(k-2)+1} \zeta_{r,k-1}^{-k+2} \eta_{r,k}^{-1} b_{r,k}^{k-1} \eta_{r,k}^{-k+2} \zeta_{r,k-1}^{-k+2} \\
\left( \begin{array}{c} \forall a \in \mathbb{Z}, a + \lambda(a) \\ = \lambda(a+1) \end{array} \right) &= \nu^{\lambda(k)} \mu^{\lambda(k-2)+1} \zeta_{r,k-1}^{-k+2} \eta_{r,k}^{-1} b_{r,k}^{k-1} \eta_{r,k}^{-k+2} \zeta_{r,k-1}^{-k+2} \\
([b_{r,k}, \eta_{r,k}] = 1) &= \nu^{\lambda(k)} \mu^{\lambda(k-2)+1} \zeta_{r,k-1}^{-k+2} b_{r,k}^{k-1} \eta_{r,k}^{-k+1} \zeta_{r,k-1}^{-k+2} \\
\left( \begin{array}{c} [\zeta_{r,k-1}, \eta_{r,k}] = \nu \\ [\zeta_{r,k-1}, b_{r,k}] = \nu \\ \mu^2 = \nu^2 = 1 \\ \forall k, (k-2)(k-1) \equiv 0 \pmod{2} \end{array} \right) &= \nu^{\lambda(k)} \mu^{\lambda(k-2)+1} b_{r,k}^{k-1} \eta_{r,k}^{-k+1}.
\end{aligned}$$

Now assume that  $a < r < b + a$ ,  $r - a + 1 \leq k \leq b$ . Denote  $k' = r - a + 1$ . So by using  $v_{r,k+1} = (v_{r,k})_{\tilde{x}_{rk}^{-1} \tilde{z}_{rk} \tilde{z}_{r+1,k}^{-1} \tilde{x}_{r+1,k+1}}$ , we see that

$$v_{rk} = \mu^{\lambda(k'-2)+1} \nu^{\lambda(k')} b_{r,k}^{k'-1} \eta_{r,k}^{-k'+1}. \tag{4.4}$$

**Remark 4.5.** (1) As in [2], we can consider a 3-point  $\omega_{r,b}$ ,  $r < a$  and see that

$$b_{11}^b \eta_{11}^{2-b} = (\mu\nu)^{\lambda(b+1)} \mu$$

(see [2, Proposition 11,(1)]).

(2) If  $b$  is odd, then  $\mu = \nu$  [2, Proposition 11,(3)].

Consider now the 3-point  $\omega_{a,b}$ . We know that  $v_{a,b} = 1$ , but  $d_{a,b} = a_{a,b}^{a-b} \zeta_{a,b}^{b-a} (\mu\nu)^{\lambda(b-a)}$ .

**Proposition 4.6.** (1) If  $a \neq b$ , then  $\mu(\mu\nu)^{\lambda(b-a+1)} = (b_{1,1} \eta_{1,1}^{-1})^{a-b} \eta_{1,1}^{-1}$ .

(2) If  $a = b$ , then  $\mu = \eta_{1,1} = 1$ .

*Proof.* (1) By Remark 4.4,

$$\begin{aligned}
 1 &= \tilde{x}_{a,b}^{-2} (a_{a,b}^{a-b} \xi_{a,b}^{b-a} (\mu\nu)^{\lambda(b-a)})_{y_{a,b}^{-1} x_{a,b}^{-1}} (\tilde{x}_{a,b}^2)_{y_{a,b}^{-1}} \\
 &\left( \begin{array}{l} \tilde{x}_{a,b}^{-2} (\tilde{x}_{a,b}^2)_{y_{a,b}^{-1}} \\ = \mu \eta_{a,b}^{-1}; \\ b_{a,b} = (a_{a,b})_{y_{a,b}^{-1} x_{a,b}^{-1}} \\ \eta_{a,b} = (\xi_{a,b})_{y_{a,b}^{-1} x_{a,b}^{-1}} \end{array} \right) = b_{a,b}^{a-b} \eta_{a,b}^{b-a} \nu^{a-b} \mu^{b-a} (\mu\nu)^{\lambda(b-a)} \mu \eta_{a,b}^{-1} \\
 &= b_{a,b}^{a-b} \eta_{a,b}^{b-a-1} (\mu\nu)^{b-a} (\mu\nu)^{\lambda(b-a)} \mu \\
 &= b_{a,b}^{a-b} \eta_{a,b}^{b-a-1} (\mu\nu)^{\lambda(b-a+1)} \mu.
 \end{aligned} \tag{4.5}$$

So  $\exists \gamma \in \tilde{B}_n$  s.t.  $(b_{a,b})_\gamma = b_{1,1}$ ,  $(\eta_{a,b})_\gamma = \eta_{1,1}$ . Applying it, we obtain what we wanted.

(2) By Remark 4.4, we have

$$1 = \tilde{x}_{a,b}^{-2} (\tilde{x}_{a,b}^2)_{y_{a,b}^{-1}} = \mu \eta_{a,b}^{-1} \tag{4.6}$$

or

$$\mu = \eta_{a,b}.$$

By the same argument as in (1),  $\mu = \eta_{1,1}$ . By (4.6), we see that  $\tilde{x}_{a,b}^2 (\tilde{x}_{a,b}^{-2})_{y_{a,b-1}} = 1$ , or  $\eta_{a,b} = 1$ ; that is,  $\mu = \eta_{1,1} = 1$ .  $\square$

**Proposition 4.7.** *If  $a \neq b$ , then  $(b_{1,1} \eta_{1,1}^{-1})^{a-b} \eta_{1,1}^{-1} = (\mu\nu)^{\lambda(b-a)}$ .*

*Proof.* By (4.5),

$$\eta_{a,b} = (a_{a,b}^{a-b} \xi_{a,b}^{b-a} (\mu\nu)^{\lambda(b-a)})_{y_{a,b}^{-1} x_{a,b}^{-1}} = b_{a,b}^{a-b} \eta_{a,b}^{b-a} (\mu\nu)^{\lambda(b-a)}.$$

Applying  $\gamma$  from above, we are done.  $\square$

Note that if  $a = b$ , we get  $b_{1,1}^b = \nu^{\lambda(b+1)}$ .

**Proposition 4.8.** *If  $b$  is even,  $a$  is odd, then  $\nu = 1$ ; otherwise  $\mu = \nu = 1$ .*

*Proof.* We will first prove a lemma.

**Lemma 4.1.**  $\forall r, a < r < a + b$ , we have

$$\eta_{1,1}^{b-a-1} b_{1,1}^{a-b} = \mu^{\lambda(r-a-1)} \nu^{\lambda(r-a+1)} (\mu\nu)^{\lambda(b-r+1)}.$$

*Proof.* By Remark 4.4 and (4.4), we have from the 3-point  $\omega_{rb}$  ( $k' = r - a + 1$ ):

$$\begin{aligned}
 \mu^{\lambda(k'-2)+1} \nu^{\lambda(k')} b_{r,b}^{k'-1} \eta_{r,b}^{-k'+1} &= \tilde{x}_{r,b}^{-2} (a_{r,b}^{r-b} \xi_{r,b}^{b-r} (\mu\nu)^{\lambda(b-r)})_{y_{r,b}^{-1} x_{r,b}^{-1}} (\tilde{x}_{r,b}^2)_{y_{r,b}^{-1}} \\
 &= b_{r,b}^{r-b} \eta_{r,b}^{b-r} \nu^{r-b} \mu^{b-r} (\mu\nu)^{\lambda(b-r)} \mu \eta_{r,b}^{-1} \\
 &\Rightarrow \mu^{\lambda(k'-2)} \nu^{\lambda(k')} b_{r,b}^{k'-1} \eta_{r,b}^{-k'+1} = \nu^{r-b} \mu^{b-r} (\mu\nu)^{\lambda(b-r)} \eta_{r,b}^{b-r-1} b_{r,b}^{r-b} \\
 &\Rightarrow \eta_{r,b}^{b-r-1+k'-1} b_{r,b}^{r-b-k'+1} = \mu^{\lambda(k'-2)+r-b} \nu^{\lambda(k')+b-r} (\mu\nu)^{\lambda(b-r)} \\
 &\stackrel{k'=r-a+1}{\Rightarrow} \eta_{r,b}^{b-a-1} b_{r,b}^{a-b} = \mu^{\lambda(r-a-1)} \nu^{\lambda(r-a+1)} (\mu\nu)^{\lambda(b-r+1)}
 \end{aligned}$$

$\forall r, \exists \gamma_r \in \tilde{B}_n$ , s.t.  $(\eta_{r,b})_{\gamma_r} = \eta_{1,1}$ ,  $(b_{r,b})_{\gamma_r} = b_{1,1}$ . Apply it, and we are done.  $\square$



Assume  $b$  is odd. So we know that  $\mu = \nu$  (By Remark 4.5). If  $a = b$ , then  $\mu = \nu = 1$  (by Proposition 4.6) Else,  $a \neq b$ . So from Lemma 4.1, set  $r = a + 1$ , and we get  $\eta(b_{1,1}\eta_{1,1}^{-1})^{a-b}\eta_{1,1}^{-1} = \mu$ . From Proposition 4.7, if  $\mu = \nu$ ,  $(b_{1,1}\eta_{1,1}^{-1})^{a-b}\eta_{1,1}^{-1} = 1$ , so  $\mu = 1 \Rightarrow \mu = \nu = 1$ .

Assume now that  $b$  is even. From Lemma 4.1, when setting  $r = a + 1$ , we get

$$(b_{1,1}\eta_{1,1}^{-1})^{a-b}\eta_{1,1}^{-1} = \nu(\mu\nu)^{\lambda(b-a)}. \quad (4.7)$$

If  $a = b$ , then  $\eta_{1,1}^{-1} = \nu$ ; but  $\eta_{1,1} = 1$ , so  $\mu = \nu = 1$ . Else ( $a \neq b$ ), we have by Proposition 4.7,  $(b_{1,1}\eta_{1,1}^{-1})^{a-b}\eta_{1,1}^{-1} = (\mu\nu)^{\lambda(b-a)}$ . So we have  $\nu = 1$  when  $b$  is even. Assume now that  $a$  is also even (and  $a \neq b$ ). By Proposition 4.6, we get

$$(b_{1,1}\eta_{1,1}^{-1})^{a-b} = \mu(\mu\nu)^{\lambda(b-a+1)} \quad (4.8)$$

or (substituting  $\nu = 1$ ), we have the set of equations:

$$\begin{cases} (b_{1,1}\eta_{1,1}^{-1})^{a-b}\eta_{1,1}^{-1} = \mu \cdot \mu^{\lambda(b-a+1)} \\ (b_{1,1}\eta_{1,1}^{-1})^{a-b}\eta_{1,1}^{-1} = \mu^{\lambda(b-a)} \end{cases}$$

Thus,

$$\mu \cdot \mu^{\lambda(b-a+1)} = \mu^{\lambda(b-a)} \Rightarrow \mu \cdot \mu^{b-a} = 1 \xrightarrow{b-a \text{ is even}} \mu = 1.$$

□

As in [2], we define a  $\tilde{B}_n$ -group  $G_0(n)$  as the subgroup of  $G(n)$  generated by  $u_1, \dots, u_{n-1}$ ;  $G_0(n)$  is  $\tilde{B}_n$ -isomorphic to  $\tilde{P}_{n,0}$  (recall that  $n = 2ab + b^2$ ).

**Definition 4.12.**  $G_0(n)$  is a group with

Generators:

$$\begin{aligned} M_0 = \{ & A_{ij}, 1 \leq j \leq b, 1 \leq i \leq a + j; \quad B_{ij}, 1 \leq j \leq b, 1 \leq i \leq a + j - 1; \\ & C_{ij}, a < i < a + b, j = i - a \}. \end{aligned}$$

Relations:

- (1)  $\forall a, b \in M_0$  which are adjacent,  $[a, b] = \tau$ , where  $\tau$  is independent of (such)  $a, b$ ,  $\tau^2 = 1$ ,  $\tau_d = \tau \forall d \in M_0$ .
- (2) If  $a, b \in M_0$  are not adjacent, then  $[a, b] = 1$ .

for each  $d \in M_0$  we introduce the notion of supporting half-twist from  $\tilde{B}_n$  (resp.  $B_n$ ) as follows: for  $d = A_{ij}$ , it will be  $x_{ij}$  (resp.  $X_{ij}$ ); for  $d = B_{ij}$ , it will be  $y_{ij}$  (resp.  $Y_{ij}$ ); for  $d = C_{ij}$ ,  $i = j + a$ , it will be  $z_{ij}$  (resp.  $Z_{ij}$ ).

We say that  $a, b \in M_0$  are adjacent if their supporting half-twists are adjacent.

The  $\tilde{B}_n$ -action on  $G_0(n)$  in terms of  $\tilde{M} = \{x_{ij}, y_{ij}\} \cup \left\{z_{ij} \mid \begin{smallmatrix} a < i < a+b \\ j=i-a \end{smallmatrix} \right\}$  and  $M_0$  is defined in [2, Remark 6]. We consider the elements of

$$\tilde{M}_1 = \tilde{M} \cup \{z_{ij} \mid (i, j) \in \text{Vertices}(K(a, b)), i, j \geq 1 \text{ and if } a < i < a + b, \text{ then } j \neq i - a\}$$

as polarized half-twists, and define a larger subset of  $G_0(n) : \hat{M}_0$ ; when  $M_0 \subset \hat{M}_0$  s.t.:

$$\hat{M}_0 = M_0 \cup \{C_{ij} \mid (i, j) \in \text{Vertices}(K(a, b)), i, j \geq 1 \text{ and if } a < i < a + b, \text{ then } j \neq i - a\}.$$

We start with the pair  $\{B_{1,1}, y_{1,1}\}$ . Then  $\forall t \in \tilde{M}_1$ , define  $L_0(t) \in \tilde{M}_0$  as the unique element  $L_{\{B_{1,1}, y_{1,1}\}}(t)$  s.t.  $\{B_{1,1}, y_{1,1}\}$  and  $\{L_{\{B_{1,1}, y_{1,1}\}}(t), t\}$  are coherent. The definition of a  $\tilde{B}_n$ -action on  $G_0(n)$  is such that  $L_0(x_{ij}) = A_{ij}$ ,  $L_0(y_{ij}) = B_{ij}$ ,  $L_0(z_{ij}) = C_{ij}$  where  $a < i < a + b$ ,  $j = i - a$ . So for  $t \in \tilde{M}$ , we have  $L(t) \in M_0$ .

Define  $C_{ij} = L_0(z_{ij})$ .

**Definition 4.13.** Using the  $\tilde{B}_n$ -action on  $G_0(n)$ , we define canonically the semi-direct product  $G_0(n) \rtimes \tilde{B}_n$ . Let  $u = y_{1,1}^2 x_{2,1}^{-2} \in \tilde{P}_{n,0} \subset \tilde{B}_n$ . Let  $N(a, b)$  be the normal subgroup of  $G_0(n) \rtimes \tilde{B}_n$ , normally generated by the elements:

$$\begin{aligned} n_1 &= B_{1,1}^b u^{2-b} c(c\tau)^{\lambda(b+1)}; \\ n_2 &= (c\tau)^b; \\ n_3 &= (B_{1,1} u^{-1})^{a-b} u^{-1} \cdot c(c\tau)^{\lambda(b-a+1)}; \\ n_4 &= (B_{1,1} u^{-1})^{a-b} u^{-1} \cdot \tau(c\tau)^{\lambda(b-a)} \end{aligned}$$

(when  $c = [x^2, y^2]$ ,  $x, y$  are any two adjacent half-twists in  $\tilde{B}_n$ ;  $\lambda(k) = \frac{k(k-1)}{2}$ ).

Note that the elements in  $N(a, b)$  are defined according to the relations found in Proposition 4.8 ((4.7), (4.8)) and Remark 4.5.

Define

$$G(a, b) = (G_0(n) \rtimes \tilde{B}_n) / N(a, b).$$

So as in [2, Proposition 32], one can prove that

$$\pi_1(\mathbb{C}^2 - S_{F_{1,(a,b)}}) \simeq G(a, b).$$

Define  $\psi_{a,b} : G(a, b) \rightarrow S_n$ , by  $\psi_{a,b}(\alpha, \beta) = \psi(\beta)$  where  $\psi : \tilde{B}_n \rightarrow S_n$  is the homomorphism to the symmetric group, induced from the standard homomorphism  $B_n \rightarrow S_n$ . Let  $H_{a,b} = \ker \psi_{a,b}$ ,  $(H_{a,b})_0 = \ker(H_{a,b} \rightarrow \text{Ab}(G(a, b)))$ , or, in other words, if  $\text{Ab}_{a,b}$  = abelization map of  $G(a, b)$ , then  $(H_{a,b})_0 = \ker \psi_{a,b} \cap \ker \text{Ab}_{a,b}$ . Note that  $G(a, b)/H_{a,b} \simeq S_n$ . Also define  $\bar{\psi}_{a,b} : \pi_1(\mathbb{CP}^2 - \bar{S}_{F_{1,(a,b)}}) \rightarrow S_n$ , and let  $\bar{H}_{a,b} = \ker \bar{\psi}_{a,b}$ . In the same way as above, we define  $(\bar{H}_{a,b})_0$  and  $(\bar{H}_{a,b})'_0$ .

So we have the following

**Theorem 4.1.**

- 1)  $H_{a,b}/(H_{a,b})_0 \simeq \mathbb{Z}$ .
- 2)  $H'_{a,b} = (H_{a,b})'_0 \simeq \begin{cases} \mathbb{Z}_2 & b \text{ even, } a \text{ odd} \\ 1 & \text{else} \end{cases}$   
 $H'_{a,b} \subset \text{Center}(G(a, b)).$
- 3)  $\text{Ab}(H_{a,b})_0 \simeq (\mathbb{Z}_{b-2a})^{n-1}.$

*Proof.* The statement can be deduced directly from the definition of  $G(a, b)$ . 2) follows from Proposition 4.8. 3) follows from the definition of  $N(a, b)$  and the following facts:

$$n_1 = B_{1,1}^b u^{2-b} c(c\tau)^{\lambda(b+1)} = (B_{1,1} u^{-1})^b u^2 c(c\tau)^{\lambda(b+1)} \text{ and}$$

$$\begin{aligned} \mathbb{Z}^2 / \langle (b, 2), (a-b, -1) \rangle &= \mathbb{Z}^2 / \langle (b, 2), (a, -1) \rangle = \mathbb{Z}^2 / \langle (b-2a, 0), (a-b, 1) \rangle \\ &= \mathbb{Z}^2 / \langle (b-2a, 0), (0, 1) \rangle = \mathbb{Z}_{b-2a}, \end{aligned}$$

□

As in [2, p. 74], one can consider the projective case

$$\pi_1(\mathbb{CP}^2 - \bar{S}_{F_{1,(a,b)}}) \simeq G(a, b) / (y_{1,0}^{2m_1} \cdot U),$$

where  $2m_1 = \deg \bar{S}_{F_1, (a,b)} = 6ab - 2a - 2b - 3b^2$ ,  $U \in (H_{a,b})_0$ . From the definition of  $\bar{H}_{a,b}$ ,  $(\bar{H}_{a,b})_0$  it follows that they coincide with the images of  $H_{a,b}$  and  $(H_{a,b})_0$  in  $G(a,b)/(y_{1,0}^{2m_1} \cdot U) = \bar{G}(a,b)$ . So by the same arguments as in [2], we have

$$\bar{H}_{a,b}/(\bar{H}_{a,b})_0 \simeq \mathbb{Z}_{m_1}, \quad (\bar{H}_{a,b})_0 \simeq (H_{a,b})_0,$$

so

$$Ab(\bar{H}_{a,b})_0 \simeq (\mathbb{Z}_{b-2a})^{n-1},$$

and

$$\bar{H}'_{a,b} \simeq (\bar{H}_{a,b})'_0 \simeq (H_{a,b})'_0 \simeq \begin{cases} \mathbb{Z}_2 & b \text{ even, } a \text{ odd} \\ 1 & \text{else} \end{cases}$$

Thus, there exists a series

$$1 \triangleleft (H_{a,b})'_0 \triangleleft (H_{a,b})_0 \triangleleft H_{a,b} \triangleleft G(a,b)$$

s.t.

$$\begin{aligned} G(a,b)/H_{a,b} &\simeq S_n \\ H_{a,b}/(H_{a,b})_0 &\simeq \mathbb{Z} \\ (H_{a,b})_0/(H_{a,b})'_0 &\simeq (\mathbb{Z}_{b-2a})^{n-1}, \end{aligned}$$

and

$$(H_{a,b})'_0 \simeq \begin{cases} \mathbb{Z}_2 & b \text{ even, } a \text{ odd} \\ 1 & \text{else} \end{cases}$$

and a series:

$$1 \triangleleft (\bar{H}_{a,b})'_0 \triangleleft (\bar{H}_{a,b})_0 \triangleleft \bar{H}_{a,b} \triangleleft \bar{G}(a,b)$$

s.t.

$$\begin{aligned} \bar{G}(a,b)/\bar{H}_{a,b} &\simeq G(a,b)/H_{a,b} \\ \bar{H}_{a,b}/(\bar{H}_{a,b})_0 &\simeq \mathbb{Z}_{m_1} \\ (\bar{H}_{a,b})_0/(\bar{H}_{a,b})'_0 &\simeq (H_{a,b})_0/(H_{a,b})'_0, \end{aligned}$$

and

$$(\bar{H}_{a,b})'_0 \simeq (H_{a,b})'_0$$

## 5. APPENDIX

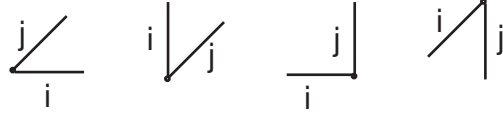
This Appendix describes the braid monodromy factorizations induced from the regeneration of each point and the induced relations from it.

For computing explicitly the braid monodromy factorizations  $\mathcal{H}(r,k)$  induced from the 6/3/2-points -  $\omega_{r,k}$ , we use the results of [2].

For  $(r,k) = (0,0), (a,0)$ , the vertex  $\omega_{r,k}$  is a 2-point on the edge  $L_j$  (a point which is on the intersection of two planes). Therefore, the braid monodromy factorization of the regenerated neighborhood of the vertex  $\omega_{r,k}$  is

$$\mathcal{H}(r,k) = Z_{j,j'}.$$

For  $(r,k)$  s.t  $\omega_{r,k}$  are on the boundary of  $P$  and  $(r,k) \neq (0,b), (a+b,b), (0,0), (a,0) - \omega_{r,k}$  is a 3-point (a point that lies on the intersection of three planes), such that locally it looks like one of the following configurations:



Consider the first and the third cases (where the line  $L_j$  is regenerated first). Then the braid monodromy factorization of the regenerated neighborhood of the vertex  $\omega_{r,k}$  is

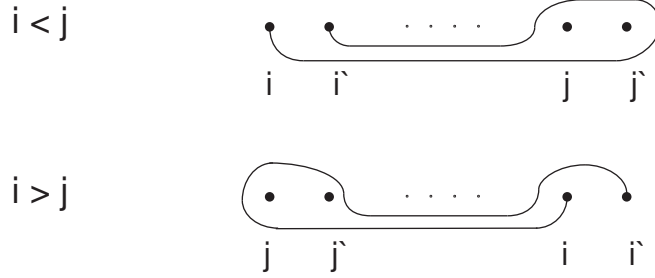
$$\mathcal{H}(r, k) = Z_{ii',j}^{(3)} \tilde{Z}_{jj'(i)}$$

when  $Z_{ii',j}^{(3)} = Z_{i',j}^3 Z_{i,j}^3 (Z_{i,j}^3)_{Z_{i,i'}}$ .

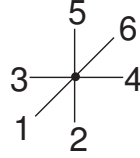
Consider the second and the fourth cases (where the line  $L_i$  is regenerated first). Then the braid monodromy factorization of the regenerated neighborhood of the vertex  $\omega_{r,k}$  is

$$\mathcal{H}(r, k) = Z_{jj',i}^{(3)} \tilde{Z}_{ii'(j)}.$$

In both cases,  $\tilde{Z}_{jj'(i)}$  is represented by the following path:



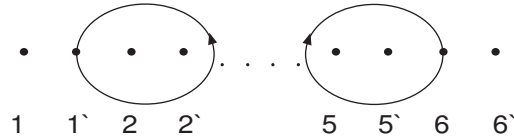
For  $(r, k)$  such that  $\omega_{r,k}$  are not on the boundary of  $P$ ,  $\omega_{r,k}$  is a 6-point. Assume that locally it looks like the following configuration (when the lines are numerated locally):



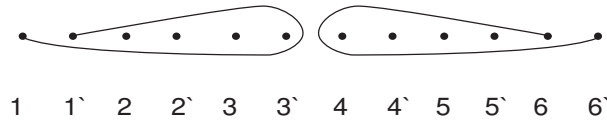
Then the braid monodromy factorization of the regenerated neighborhood of the vertex  $\omega_{r,k}$  is:

$$\mathcal{H}(r, k) = Z_{1',2,2'}^{(3)} \tilde{Z}_{6,6'} Z_{3,3',6'}^{(2)} (Z_{2,2',6'}^{(2)})^\bullet \bar{Z}_{4,4',6}^{(3)} (Z_{3,3',6}^{(2)})^\bullet (Z_{2,2',6}^{(2)})^\bullet (\hat{F}(\hat{F})_{\rho^{-1}})^\bullet Z_{5,5',6}^{(3)} \\ \left( \prod_{\substack{i=6',6,5' \\ 5,4',4}} (Z_{1',i}^2) \right)^\bullet \bar{Z}_{1',3,3'}^{(3)} \prod_{\substack{i=6',6,5' \\ 5,4',4}} (Z_{1,i}^2) \tilde{Z}_{1,1'},$$

where  $Z_{ii',j}^{(2)} = Z_{i',j}^2 Z_{i,j}^2, ()^\bullet$  is the conjugation by the braid induced from the motion:



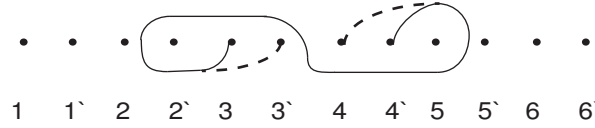
and  $\tilde{Z}_{1,1'}, \tilde{Z}_{6,6'}$  are



$$\rho = Z_{22'} Z_{55'}$$

$$\hat{F} = Z_{2',33'}^{(3)} Z_{44',5}^{(3)} \check{Z}_{3'4}^{(3-3')} Z_{2',5}^2 \bar{Z}_{2',5'}^2$$

where  $\check{Z}_{3'4}$ ,  $\bar{Z}_{3'4}$  are:



By the Van-Kampen Theorem [16], we can see that we get a triple relation  $(\langle A, B \rangle = e)$  for each pair of generators whose corresponding lines (from which they are created) induce a common triangle in the complex  $K(a, b)$ ; and we get a double (commutation) relation  $([A, B] = e)$  for each pair of generators whose corresponding lines does not induce a common triangle in the complex. This is the basis for the embedding of  $\hat{B}_n$  in  $G$ . For more details, see [2].

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